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Period doubling of a torus: Chaotic breathing of a localized wave

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This study identifies the existence of a novel route to chaos from a fixed point, to a limit cycle, to a torus, and then a cascade of period doubling of the torus, which has been predicted theoretically. This route to chaos has been found in the destabilization of a solitonlike structure present in a continuous dissipative medium. [S1063-651X(97)11803-7]

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Much effort has been devoted to the understanding of chaotic behavior in dissipative dynamical systems in recent years. One of the questions of great interest lately concerns a transition to chaos through a torus, i.e., a dynamical motion characterized by two incommensurate frequencies. As one way a torus can be destabilized, many researchers have noticed that a torus is destabilized by successive period doubling (often referred to as torus doubling). Such an investigation has been based on the study of the abstract low-dimensional iterative mappings. The mappings model a discrete flow obtained by the method of Poincaré surface of section, from a continuous flow governed by differential equations.

Kaneko [1] found a torus doubling in the study of three- and four-dimensional dissipative mappings. In the seven-mode study of the Navier-Stokes equations, Franceschini [2] observed two successive torus doublings, followed by a strange attractor. Arnéodo *et al.* [3] also found that a torus may undergo a few doublings before being replaced by a strange attractor in their investigation of three-dimensional dissipative mappings. Significantly, their three-dimensional model could show a cascade of an infinite number of torus doublings in the transition into a strange attractor. This discovery implies that the same dynamic behavior may also exist in real physical systems. Experimentally, the phenomenon of the torus doubling was observed in the Rayleigh-Bénard convection [4,5], in the convection in molten gallium [6], and in the electrochemical reactions [7].

Up to now, theoretical investigations of torus doubling have been focused on abstract low-dimensional model equations mainly because it is very difficult to find a low-dimensional subspace confining a chaotic attractor for an infinite-dimensional dynamical system. In our investigations of destabilization mechanisms of localized structures in continuous media, we could actually identify the existence of the cascade of torus doubling in a localized structure present in the quintic complex Ginzburg-Landau equation [8,9]:

$$\frac{\partial\psi}{\partial t}=\mu\psi+(\alpha+\iota)\frac{\partial^2\psi}{\partial x^2}+(\beta+2\iota)|\psi|^2\psi+(\gamma+\iota\delta)|\psi|^4\psi, \tag{1}$$

where μ, α, β, γ, and δ are all real. μ measures the distance from criticality. Equation (1) describes the dynamics of the two-dimensional disturbances of the plane Poiseuille flow with ψ representing the complex amplitude of Tollmien-Schlichting waves [10], and also the dynamics of the traveling waves occurring in binary-fluid-mixtures convection [8,9].

For the set of parameter values, $\mu=-0.15$, $\beta=1.5$, $\gamma=-1.0$, and $\delta=2.0$, the system admits a solitonlike structure. Figure 1(a) shows the localized wave obtained by integrating Eq. (1) numerically. This localized wave is created by perturbing the state $\psi=0$ with a Gaussian shape wave of a sufficient amplitude. If the perturbing amplitude is too small, the localized wave damps to zero. If the coefficients of Eq. (1) are real, the system becomes purely dissipative, and the equation can be put in a variational form which gives a solution of unstable localized waves. However, if the coefficients are complex, as they are here, localized waves can be

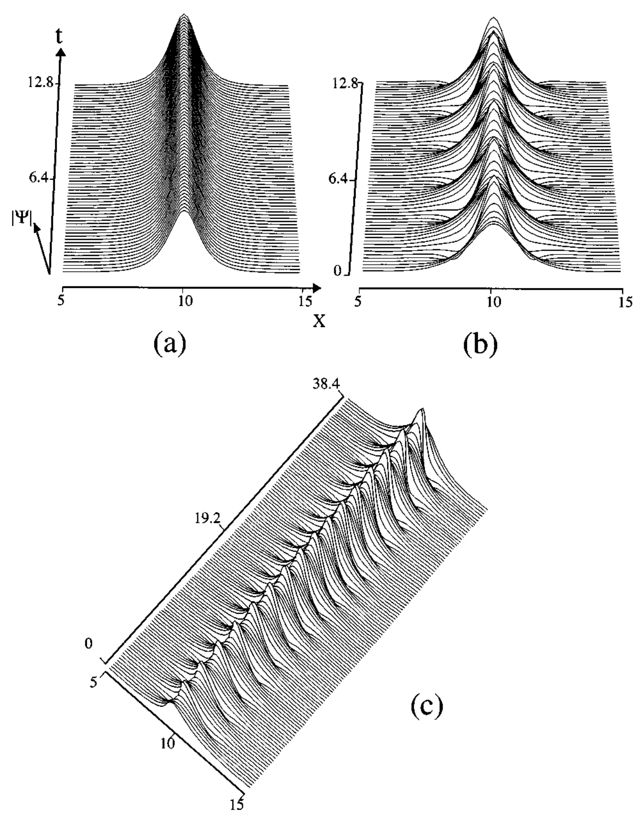

FIG. 1. Destabilization of a solitonlike coherent wave. (a) A solitonlike pulse ($\alpha=0.25$). (b) Periodic breathing of the pulse ($\alpha=0.23$). (c) Chaotic breathing of the pulse ($\alpha=0.2201$).

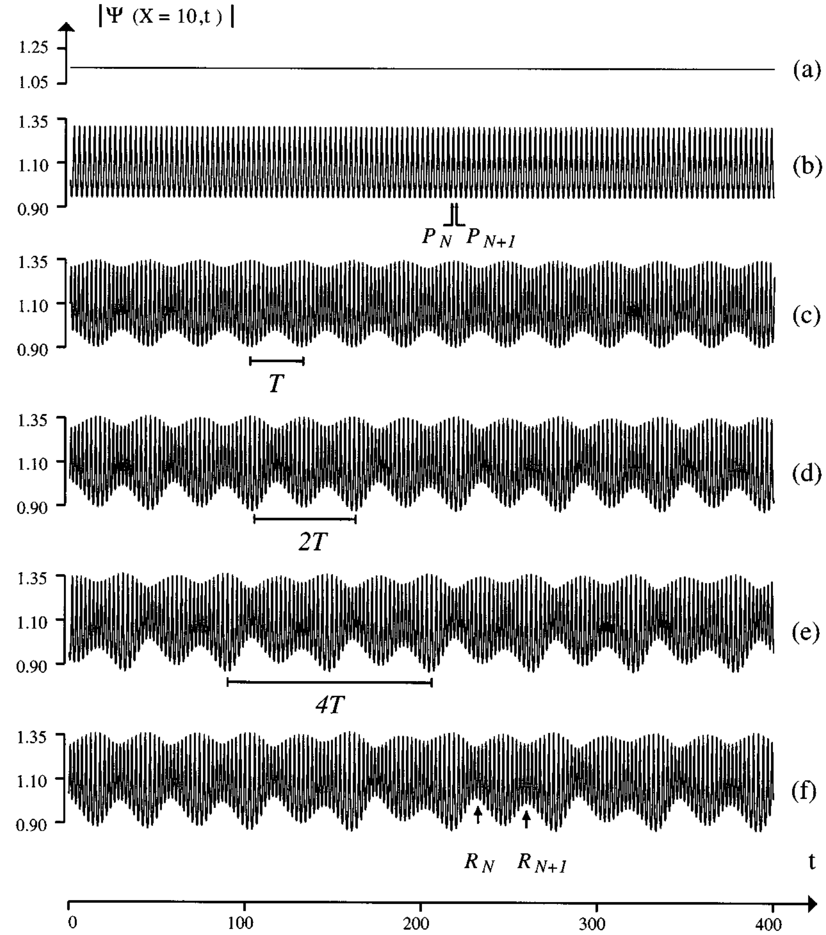

FIG. 2. Time series of the maximum amplitude of the pulses. The time series are obtained at a fixed spatial fixed point ($x=10$). (a) Fixed amplitude for the solitonlike pulse ($\alpha=0.25$). (b) Periodic oscillation with ω_1 ($\alpha=0.23$). (c) Quasiperiodic oscillation (torus) with ω_1 and $\Delta\omega=2\pi/T$ ($\alpha=0.225$). (d) Quasiperiodic oscillation with ω_1 and $\Delta\omega/2$: period doubling of the torus ($\alpha=0.2215$). (e) Secondary period doubling of the torus ($\alpha=0.2206$). (f) Chaotic oscillation ($\alpha=0.2201$).

stablized in the vicinity of a subcritical instability.

We found the cascade of torus doubling from this solitonlike structure. As α is reduced, the solitonlike structure starts to be destabilized. The amplitude of the solitonlike structure undergoes from a fixed state to a periodic state with frequency ω_1 to a quasiperiodic state with two incommensurate frequencies, ω_1, ω_2, and then a cascade of period doubling in the beat (modulation) frequency $\Delta\omega=\omega_1-\omega_2$. Eventually, we observe chaotic breathing of the localized structure. The periodic breathing of the wave and the chaotic breathing of the wave are displayed, respectively, in Fig. 1(b) and Fig. 1(c). Notice that the waves remain localized with time.

Since the localized waves maintain spatial coherence, their dynamics can be described by the time series of their maximum amplitude. In Fig. 2, we displayed the time series of the amplitude of the waves for various values of α, where α decreases from top to bottom. When α is high ($\alpha=0.25$), the amplitude of the solitonlike structure becomes stabilized and fixed. As α is reduced ($0.225<\alpha<0.25$), the amplitude starts to oscillate with a single frequency, say, ω_1 [Fig. 1(b)]. The corresponding power spectrum in Fig. 3(a) shows this frequency. As α is reduced further, the amplitude shows an additional oscillatory motion with a larger period denoted by T in Fig. 2(c), whose frequency is denoted by $\Delta\omega$ in the corresponding power spectrum shown in Fig. 3(b). This amplitude modulation frequency $\Delta\omega$ can be considered as the beat frequency of the two frequencies ω_1 and ω_2: Since ω_1 and ω_2 are independent, ω_1 and $\Delta\omega$ are independent. We shall denote the frequencies ω_1 and ω_2 as ''carrier frequencies'' and the frequency $\Delta\omega$ as a ''beat frequency.'' The beat frequency $\Delta\omega$ corresponds to the frequency of the slow time scale amplitude modulation with period T in Fig. 2(c).

In the following time series shown in Fig. 2(d), we found that the period of the slow time scale amplitude modulation has doubled, becoming $2T$. The corresponding frequency has halved as $\Delta\omega/2$, and all other frequencies present in the power spectrum can be expressed as a linear combination of two independent frequencies, ω_1 and $\Delta\omega/2$. For example, the ω_3 lying in between ω_1 and ω_2 satisfies $\omega_3=\omega_1-\Delta\omega/2$. For this state, we may consider again ω_1 and ω_3 as two independent frequencies and $\Delta\omega/2$ as their beat frequency, $\Delta\omega/2=\omega_1-\omega_3$. This way, we find that the beat frequency is halved.

In the following state, we find that the beat frequency is further halved, and more frequencies are generated in the power spectrum as a result of the linear combination of ω_1 and $\Delta\omega/4$. The power spctrum shows clearly how the new frequencies are generated.

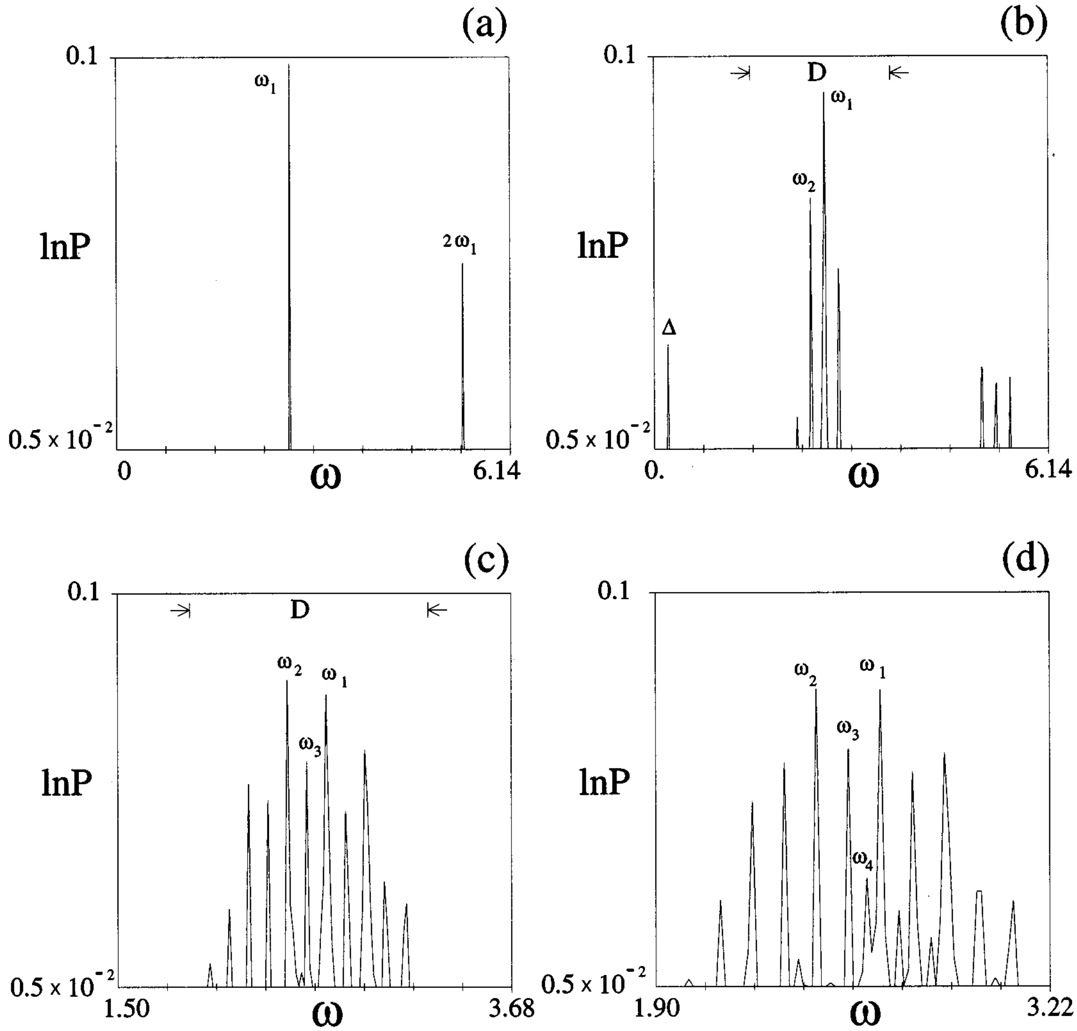

FIG. 3. Power spectra of the time series of Fig. 2. (a) $\alpha=0.23$. (b) Quasiperiodic state with ω_1 and ω_2, $\Delta\omega=\omega_1-\omega_2$ ($\alpha=0.225$). (c) Enlargement of the range denoted by ''D'' in (b) when $\alpha=0.2215$. $\omega_3=\omega_1-\Delta\omega/4$. (d) Further enlargement of the range denoted by ''D'' in (c) when $\alpha=0.2206$.

To see whether the period doubling continues in the slow time scale, we analyzed again the time series displayed in Fig. 2, but this time, we observed the time series stroboscopically at the fast frequency ω_1. Starting from any one of the extrema, we then obtained a series of discrete data, P_N, $N=1,2,\ldots,\infty$, for each time series where P_N's are local extrema as indicated in Fig. 2. To search for any order that might exist between successive outcomes P_N and P_{N+1}, we have plotted P_{N+1} as a function of P_N for all the time series shown in Fig. 2. The results are shown in Fig. 4. In the case of the single frequency motion with ω_1, shown in Fig. 2(d), for which the resulting P_N's are all the same, its $P_{N+1}-P_N$ plot shows only one fixed point denoted by ''O'' in Fig. 4(a).

For the biperiodic motion with two frequencies ω_1 and $\Delta\omega=2\pi/T$, its $P_{N+1}-P_N$ plot removes the frequency ω_1 and gives a trajectory making a closed loop circular motion around the fixed point ''O'' with period T. So a circular closed orbit on the $P_{N+1}-P_N$ plane indicates that the motion is biperiodic and the fact that the trajectory completely fills up the loop indicates that the motion is indeed quasiperiodic.

Period doubling of the torus, as α decreases, is clearly portrayed in Fig. 4. From this observation, we can expect that the nth subharmonics ω_n are excited successively at the left side of the ω_1 at a distance $\Delta\omega/2^{n-2}$ (or $(\omega_1-\omega_2)/2^{n-2}$) where $n\geq 3$. Eventually, as illustrated in Fig. 4(e), the trajectory forms circular bands. The merging of the bands, which resembles the reverse bifurcations of the familiar theory of period doubling, is depicted in Fig. 4(f). The evolution of the trajectories displayed in Fig. 4 is believed to be enough evidence for the existence of a cascade of period doubling of a torus. Actually the similar plots as shown here are what were observed in the study of the three-dimensional mappings by Arnéodo *et al.* and became the theoretical basis for the possible existence of the cascade of period doubling of a torus.

Next, we want to confirm that the final plot in Fig. 4(f) indeed describes a chaotic behavior. For that purpose, we further reduced the dynamics into an iterative one-dimensional mapping. In the corresponding time series shown in Fig. 2(f), we considered the slow time scale extrema, denoted by R_N, $N=1,2,\ldots,\infty$. From this set of data,

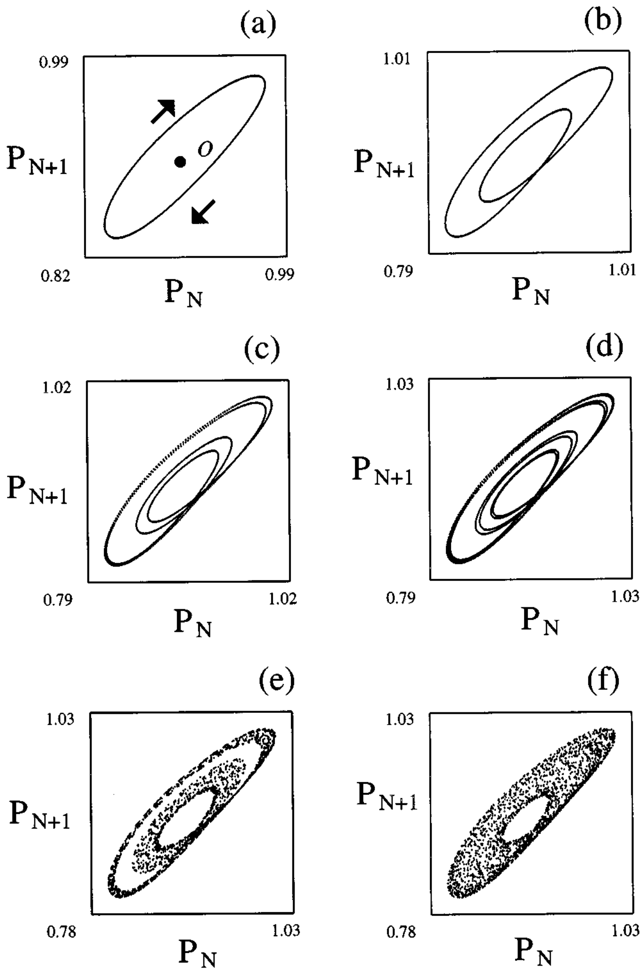

FIG. 4. P_{N+1} as a function of P_N, $N=1,2,\ldots,\infty$ (cf. Fig. 2). P_N is the nth extremum of the amplitude of the pulse. (a) The fixed point "O" represents the periodically oscillating amplitude with frequency ω_1. The closed orbit around "O" represents a torus corresponding to the quasiperiodic state of Fig. 2(c). The period of this closed orbit is T. (b) Period doubling of the torus ($\alpha=0.2215$). (c) $\alpha=0.2206$. (d) $\alpha=0.220\,46$. (e) $\alpha=0.2203$. (f) $\alpha=0.2201$.

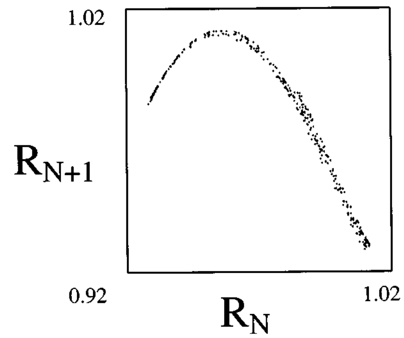

FIG. 5. Return map constructed from the slow-time scale extrema R_N, $N=1,2,\ldots,\infty$, for the chaotically breathing state [cf. Fig. 2(f)].

we constructed a return map R_{N+1} as a function R_N, which is shown in Fig. 5. Since the map is concave down, and has a smooth maximum, it is a quadratic type map. It is well established that this type of map indeed describes a chaotic behavior. The thickness in the map results from the discrepancy in the maximum values of the discrete P_N and the true maxima of the continuous envelope of the localized wave. But the map still gives a definite form of the quadratic map, and therefore gives concrete evidence that the behavior is indeed chaotic. Indeed this type of map has been the basis for Feigenbaum's period doubling theory. This extraction of a quadratic-type map from the cascade of the torus doubling implies the broad applicability of Feigenbaum's period doubling theory.

We finally point out that some theories [1,11,12] and most experiments reported that torus doubling cascade would be difficult to observe. But, recent experiments in Rayleigh-Bénard convection [5] have shown that the whole scenario of a torus doubling does exist.

In conclusion, we have identified the existence of a novel route to chaos, from a fixed point, to a limit cycle, to a torus, and then a cascade of period doubling of the torus, which has been predicted theoretically. Significantly, we found the cascade of torus doubling in the destabilization of a localized structure in a continuous medium.

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